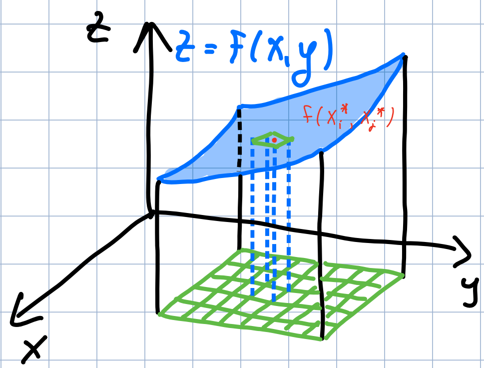
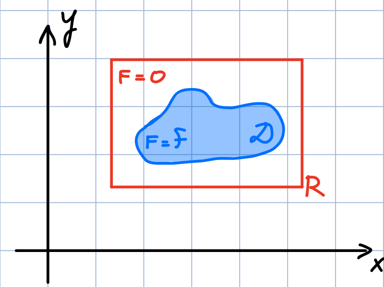
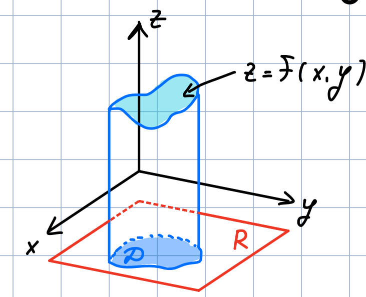
Last time

$$\iint_R f(x,y)\,dA = \lim_{\substack{m\to\infty\\ n\to\infty}} \sum_{i,j} f(x_i^*,y_j^*)\underbrace{\Delta A}_{\Delta x\cdot\Delta y} \overset{\text{for } f\geq 0}{=}$$ volume of a solid that lies above the rectangle (general region) and below the surface $z=f(x,y)$.



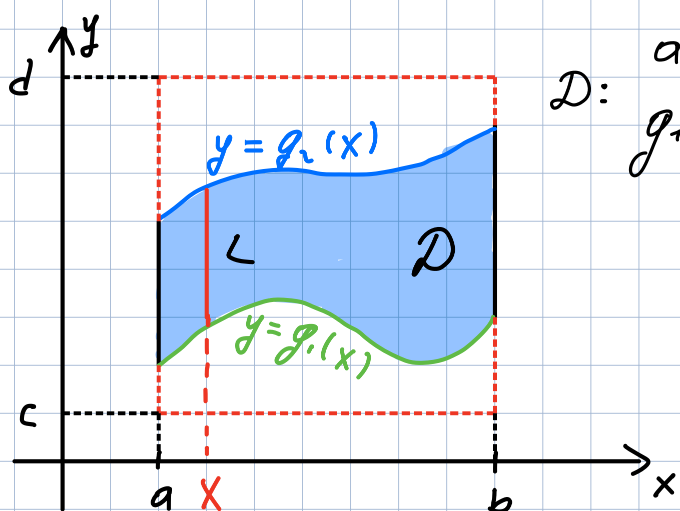
(x_i^*, y_j^*) – sample point in $[x_{i-1},x_i]\times[y_{j-1},y_j]$





Fubini's thm: if $f(x,y)$ continuous

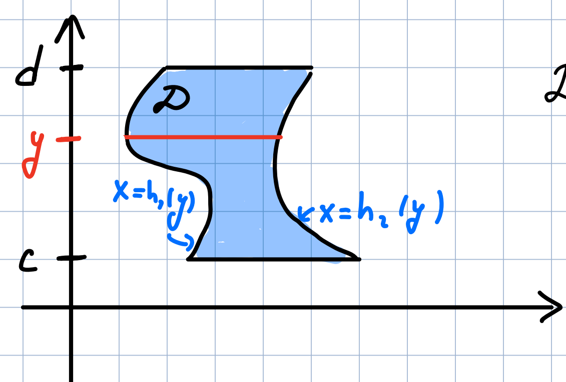
$$\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$



D: $a\leq x\leq b$
$g_1(x)\leq y\leq g_2(x)$

region of type I

$$\boxed{\iint_D f(x,y)\,dA = \int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx}$$

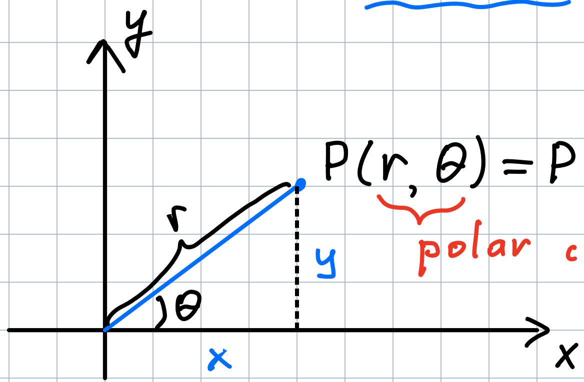


D: $c\leq y\leq d$
$h_1(y)\leq x\leq h_2(y)$

region of type II

$$\boxed{\iint_D f(x,y)\,dA = \int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy}$$

Double integrals in polar coordinates



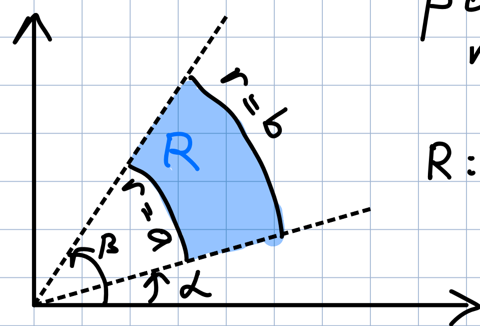
$P(\underbrace{r,\theta}_{\text{polar coordinates}}) = P(x, y)$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Assume: $0 \le a \le b$, $0 \le \beta - \alpha \le 2\pi$

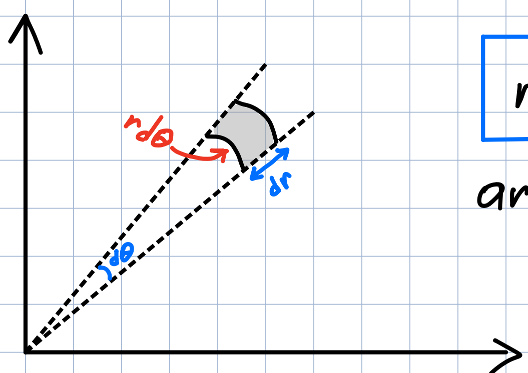
polar rectangle



$R: \begin{array}{l} a \le r \le b \\ \alpha \le \theta \le \beta \end{array}$

don't forget this factor!

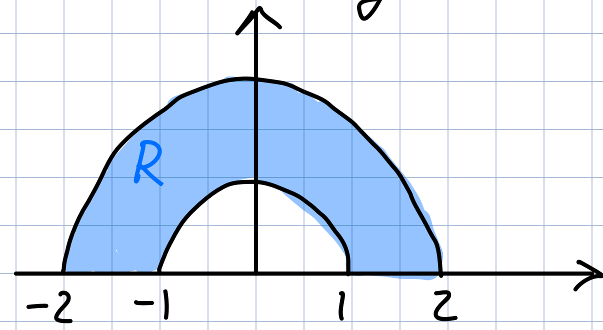
$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,\underbrace{\underline{r}\,dr\,d\theta}$$



$\boxed{r\,dr\,d\theta}$

area of an infinitesimal polar rectangle

Ex: Find $\iint\limits_R (3x+4y^2)\,dA$, where R is the region in upper half-plane bounded by circles $x^2+y^2=1$ and $x^2+y^2=4$.



Sol:

$$R=\left\{(x,y)\ \middle|\ \begin{array}{l} y\geq 0 \\ 1\leq x^2+y^2\leq 4\end{array}\right\}$$

$$=\left\{(r,\theta)\ \middle|\ \begin{array}{l} 1\leq r\leq 2 \\ 0\leq \theta\leq \pi\end{array}\right\} \text{ - polar rectangle}$$

$$\iint\limits_R (3x+4y^2)\,dA=\int_0^{\pi}\int_1^2 (3r\cos\theta+4r^2\sin^2\theta)\,r\,dr\,d\theta$$

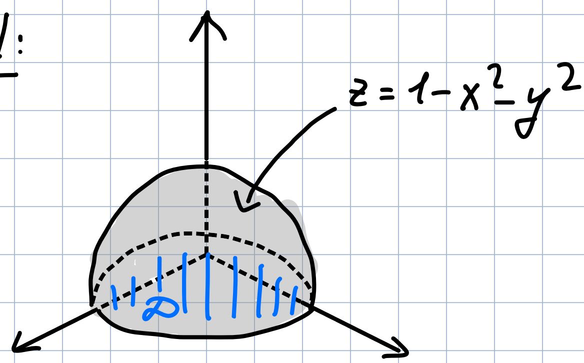
$$=\int_0^{\pi}\underbrace{\int_1^2 (3r^2\cos\theta+4r^3\sin^2\theta)\,dr}_{r^3\cos\theta+r^4\sin^2\theta\Big|_{r=1}^{r=2}=7\cos\theta+15\sin^2\theta}\,d\theta$$

$$=\int_0^{\pi}(7\cos\theta+15\underbrace{\sin^2\theta}_{\frac{1-\cos 2\theta}{2}})\,d\theta$$

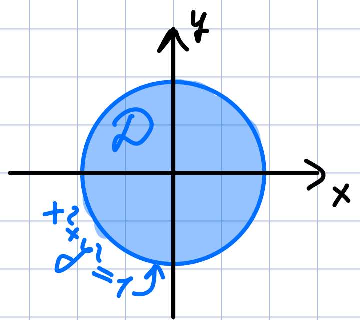
$$=7\sin\theta+\frac{15}{2}\theta-\frac{15}{2}\cdot\frac{1}{2}\cdot\sin 2\theta\Big|_0^{\pi}=\frac{15}{2}\pi$$

Ex: Find the volume V of the solid bounded by the plane $z=0$ and paraboloid $z=1-x^2-y^2$

Sol:



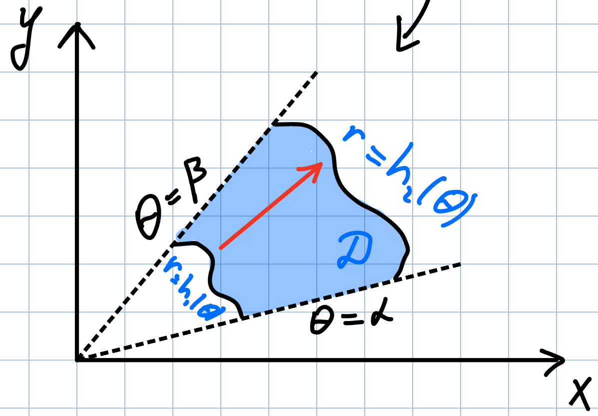
the solid lies over the disc $x^2+y^2\le 1$ in xy-plane and below surface $z=1-x^2-y^2$



$$V=\iint_{D}(1-x^2-y^2)\,dA \underset{\text{polar}}{=} \int_0^{2\pi}\underbrace{\int_0^1 (1-r^2)\,r\,dr}_{\frac{1}{2}r^2-\frac{1}{4}r^4\Big|_{r=0}^{r=1}=\frac{1}{4}}\,d\theta$$

$$=\int_0^{2\pi}\frac{1}{4}\,d\theta = 2\pi\cdot\frac{1}{4}=\frac{\pi}{2}$$

For $\mathcal{D}$ like that

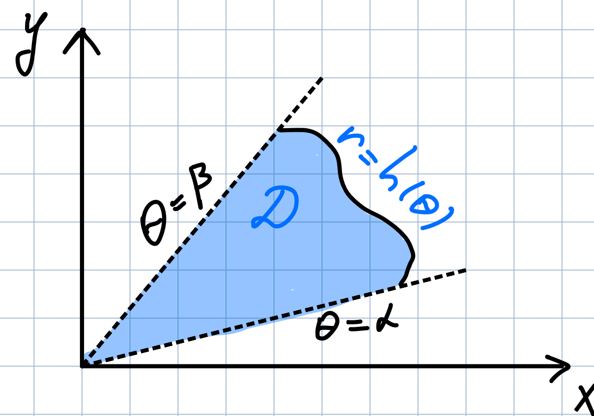


$$\mathcal{D}=\left\{(r,\theta)\;\middle|\;\begin{array}{l}\alpha\le\theta\le\beta\\ h_1(\theta)\le r\le h_2(\theta)\end{array}\right\}$$

$$\iint_{\mathcal{D}} f(x,y)\,dA=\int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

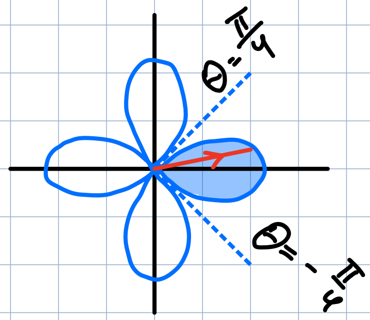
- Special case: $f(x,y)=1$, $h_1(\theta)=0$, $h_2(\theta)=h(\theta)$.

$$\underbrace{\iint_{\mathcal{D}} 1\,dA}_{=\text{Area}(\mathcal{D})}=\int_{\alpha}^{\beta}\underbrace{\int_{0}^{h(\theta)} 1\cdot r\,dr}_{\frac{1}{2}r^2\big|_{r=0}^{r=h(\theta)}}\,d\theta=\int_{\alpha}^{\beta}\frac{h^2(\theta)}{2}\,d\theta$$



Ex: Find the area enclosed by one loop of the "4-leaved rose" $r = \cos 2\theta$.

Sol:



$$\mathcal{D} = \left\{ (r,\theta) \;\middle|\; \begin{array}{l} -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \\ 0 \le r \le \cos 2\theta \end{array} \right\}$$

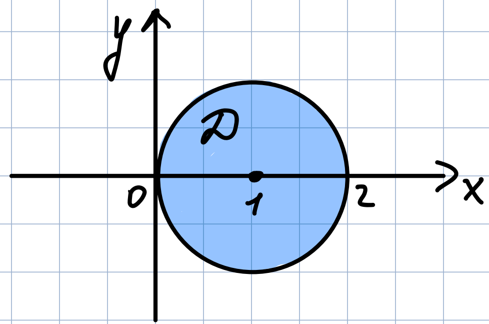
$$\text{Area}(\mathcal{D}) = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} \underbrace{\cos^2 2\theta}_{\frac{1+\cos 4\theta}{2}} \, d\theta = \frac{1}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1+\cos 4\theta)\, d\theta$$

$$= \frac{1}{4} \left(\theta + \frac{1}{4} \sin 4\theta \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \right) = \frac{\pi}{8}$$

Ex: Find the volume V of the solid under $z = x^2+y^2$, above xy-plane and inside the cylinder $x^2+y^2 = 2x$.

Sol: $x^2+y^2 = 2x \iff (x-1)^2+y^2 = 1$

Solid lies above the disc $\mathcal{D}$:



Boundary circle in polar coordinates:

$$(r\cos\theta)^2+(r\sin\theta)^2 = 2\cdot r\cos\theta \implies$$

$$r^2 = 2r\cos\theta \implies r = 2\cos\theta$$

So, $\mathcal{D} = \left\{ (r,\theta) \,\middle|\, \begin{array}{l} -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 2\cos\theta \end{array} \right\}$

$$V = \iint_{\mathcal{D}} (x^2+y^2)\,dA = \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \underbrace{\left((r\cos\theta)^2+(r\sin\theta)^2\right)}_{r^2} r\,dr}_{\frac{1}{4}r^4\Big|_{r=0}^{r=2\cos\theta} = 4\cos^4\theta}\,d\theta = 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\,d\theta$$

$$// \cos^4\theta = \left(\frac{1+\cos 2\theta}{2}\right)^2 = \frac{1}{4}+\frac{1}{2}\cos 2\theta+\frac{1}{4}(\cos 2\theta)^2 = \frac{1}{4}+\frac{1}{2}\cos 2\theta+\frac{1}{4}\left(\frac{1+\cos 4\theta}{2}\right) //$$

$$= 4\int_{-\pi/2}^{\pi/2} \left(\frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{8} + \frac{1}{8}\cos 4\theta \right) d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta \right) d\theta = \frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{3}{2}\pi$$